6 Colour Patterns

Themes	Rotational transformation and symmetry of the octahedron in three dimensions.
Vocabulary	Plane of reflection, axis of rotation, perpendicular, face, edge, vertex.
Synopsis	Investigate which polyhedra can be made with a fixed set of colours of face and different rules for how the colours can coincide at edges or vertices. Symmetries of the octahedron are related to colour-swapping transformations.

Overall structure	**Previous**	**Extension**
1 Use, Safety and the Rhombus	**X**	
2 Strips and Tunnels		
3 Pyramids	**X**	
4 Regular Polyhedra		
5 Symmetry (relates to colour-swapping transformations)		**X**
6 Colour Patterns		
7 Space Fillers		
8 Double edge length tetrahedron		
9 Stella Octangula (applies colour patterns)		**X**
10 Stellated Polyhedra and Duality		
11 Faces and Edges		
12 Angle Deficit		
13 Torus		

Layout

The activity description is in this font, with possible speech or actions as follows:

Suggested instructor speech is shown here with

possible student responses shown here.
'Alternative responses are shown in quotation marks'.

1 Building the 2 coloured octahedron

Break the class into groups of 4 or 5 students. Have each group pick two colours and take 4 to 6 triangles of each colour. Note that no net is given. Students will just build the shape up face by face. To start them off, take two triangles of different colours and join them along an edge. Then holding up the resulting rhombus point to the edge and say:

This edge has one colour on one side and another colour on the other side.

In your groups, make a closed up shape with your triangles using two colours so that every edge has one colour on one side and the other colour on the other side. There will be no edges with the same colour on both sides.

As you go around while students build (e.g. as in figure 1), remind them of the task and ask:

How many triangles did you put at this vertex?

4

Could you have made a vertex with three triangles meeting at the vertex?

No, the colours won't work

Figure 1 Building up towards the 2-coloured octahedron

Figure 2 The 2-coloured octahedron

The polyhedron with the least triangles that can be made that follows this rule is the octahedron.

2 Analysis of effects of colouring rule

Guide the discussion toward the need for an even number of triangles at every vertex, so the colours can alternate going around the vertex. This may start as above when you go round the groups and then come together in a class discussion.

In class discussion ask:

What happens if there are only three triangles at a vertex?

'Two colours the same have to go together'
'You start with one colour, switch it for the next triangle, switch again for the third, but then the first and third are the same colour coming together.

Figure 3 A vertex with 5 triangles cannot be two coloured.

What about with five?

No

Use some triangles assembled around a vertex as in figure 3 to demonstrate.

When can it work?

It has to be an even number

3 Colour swapping rotations holding the octahedron on a vertex

Now hold the octahedron upright so one vertex points up and the opposite one is on the floor. Ask the following question about transformations in 3 dimensions, this can introduce or consolidate vocabulary and concepts from the symmetry activity.

How do I need to move this shape to swap the colours, but with the shape ending in the same in same place? So after it moves there will be a vertex in all the places there was a vertex before, a face in all the places there was a face before, and an edge in all the places there was an edge before. But after it has moved, the face colours will be switched.

Turn it

Which way? Come and move it.

[A students may rotate about the vertical axis by 90 degrees]

'What is the name for this transformation?'
'What kind of movement did we just do?'

'Rotation', 'twist'

How many degrees?

90

About which axis?

Vertical

What happens if we repeat the rotation?

The colours switch back.

Can we swap the colours rotating about a different axis?

'Spin it the other way'

Show us.

Make sure both the horizontal axes through opposite vertices are tried.

Where does the axis go through the shape?

Horizontally through vertices.

Which vertices

Opposite

Do the vertices have to be opposite?

Yes

Have we found all the pairs of opposite vertices? How many are there?

Yes, 3

This discussion can continue as desired. It is possible to investigate repeated 90 degree rotations, to see how they swap colours repeatedly. It is possible to introduce, as described below, a convenient terminology for colour-preserving and colour-swapping rotations.

Further issues include investigating the geometric issues such as perpendicularity of lines connecting opposite vertices, where they meet in the centre, or the three square cross sections containing four vertices of the octahedron which are also mutually perpendicular.

4 Colour-swapping rotations holding the octahedron on an edge.

Now hold the octahedron hanging down under its own weight from the midpoint of an edge, with one edge on the floor. Repeat the questions.

How can I move this to swap the colours now, but with the shape ending up in the same place?

Turn it

Which way? Come and move it.

[A students may rotate about the vertical axis by 180 degrees]

'What is the name for this transformation?'
'What kind of movement did we just do?'

'Rotation', 'twist'

How many degrees?

180

About which axis?

Vertical

What happens if we repeat the rotation?

The colours switch back.

5 Colour swapping rotations holding the octahedron on a face.

Now place the octahedron on a face as in figure 2.

What happens now? Can we swap the colours by rotating in a vertical axis?

No

Let's try it.

Rotate the shape about the vertical axis 120 degrees.

What happened?

It did not change the colours.

So we will call this a colour-preserving rotation, and the rotations that swap colours we will call colour-swapping rotations.

6 Reflections of the 2-coloured octahedron (advanced visualisation and counting)

It is possible to consider reflections of the octahedron. These reflections may or may not swap colours. This is a high level exercise in visualisation because the reflections in 3 dimensions cannot be fully demonstrated.

If the octahedron is held resting on a vertex, it is possible to indicate a vertical plane passing through four vertices and say:

Imagine there is a mirror (indicating where it is by waving an arm). Each face on one side has a mirror image face on the other side. But, it is not exactly a mirror image because the corresponding faces on opposite sides of the mirror are different colours.

Now indicate the horizontal plane through the centre of the octahedron and say:

It is like a lake, or a horizontal mirror. Ignoring colour, do we get each face having a mirror image face?

Yes

Is each face the same colour as its mirror image face?

No

We will call these planes of reflection, 'colour-swapping planes of reflection'.

Can anyone see any more planes of reflection?

This may require you to demonstrate the plane going through the top and bottom vertex and midpoints of opposite horizontal edges. Ask:

Are the colours one side of the mirror swapped in the mirror image?

No

This is not a colour-swapping plane of reflection so we will call it a 'colour-preserving plane of reflection'.

What differences are there between a colour-swapping and colour-preserving plane of reflection?

Colour-swapping goes along edges

Colour-preserving goes through faces

Why do you think that is?

'If you reflect across an edge then a face is adjacent to its mirror image, and they must be different colours.'
'If you reflect one side of a face to the other, then the colour stays the same.'

7 Counting colour-preserving planes of reflection (advanced)

Is it possible to count how many colour-preserving planes of reflection there are? If students do not find the following counting method by themselves, then give them these hints.

Which points on the octahedron does the colour-preserving plane of reflection pass through?

Vertices and edges

Where does it pass through the edges?

At midpoints

How many midpoints?

2

Are there any other colour-preserving planes of reflection?

No

Any ideas how to use this to count the colour-preserving planes of reflection?

Continue hinting if necessary:

How many edges are there?

12

If each plane goes through 2 edges, how many planes must there be?

6

Can someone show us the six colour-preserving planes of reflection on this shape?

If you wish to be more rigorous, it is necessary to verify both firstly that:

> **For every colour-preserving plane there is just one corresponding pair of midpoints of opposite edges.**

and secondly that:

> **For every pair of midpoints of opposite edges there is just one corresponding colour-preserving plane of reflection.**

To demonstrate the need for this one to one correspondence repeat the method using vertices instead of edges.

Each plane passes through 2 vertices. How many vertices are there?

6

Wouldn't that mean there are only 3 colour-preserving planes?

Yes

Why are we getting the wrong answer?

If necessary explain, holding the octahedron on a vertex, and showing where the planes are:

We know these two vertical planes are both colour-preserving planes of reflection. Which vertices do they pass through?

The same ones

Does that explain anything?

Yes we counted pairs of vertices but 2 planes go through each pair of vertices which is why we got 3 instead of 6

When we counted pairs of midpoints of opposite edges, how do we know we got the right answer?

Because only one colour-preserving plane of reflection goes through each pair of midpoints of opposite edges. So the number of pairs of midpoints is the same as the number of planes

Yes we showed a one to one correspondence between colour-preserving planes of reflection and the pairs of midpoints of opposite edges.

Firstly,

for every colour-preserving plane there is just one pair of midpoints of opposite edges,

and secondly,

for every pair of midpoints of opposite edges there is just one colour-preserving plane of reflection.

8 Other colour rules

With four colours we use the rule that there must be one triangle of each colour at a vertex. This gives a 4-coloured octahedron, using two triangles of each colour. With five colours, we also specify one triangle of each colour at every vertex. This will give an icosahedron, with four triangles of each colour. You can allow the students to work in groups with enough or even surplus triangles to construct the shapes.

Note: These constructions do not use a net to make the shape. Following the colouring rule forces the shape to be built the right way.

9 Additional and advanced observations

Pairs of opposite faces of an octahedron sometimes have the same colour. This is true for the octahedron with 2 or 4 colours.

If a 2-coloured octahedron is stellated just on faces of one colour then the result is a tetrahedron, see activity **10 Stellated Polyhedra and Duality** for how to stellate.

There are inscribed tetrahedra in these shapes with their vertices at the midpoints of faces of the same colour, when four triangles of each colour are used. This is true both the octahedron with 2 colours and icosahedron with 5 colours.

